

PERIODICAL PLANE PUZZLES WITH NUMBERS

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1. INTRODUCTION

Consider a periodical (in two independent directions) tiling of the plane with polygons (faces). In this article we shall only give examples using squares, regular hexagons, equilateral triangles and parallelograms (“unions” of two equilateral triangles). We shall call some “multiple” of the fundamental region “the board”. We naturally identify pairs of corresponding edges of the the board. Figures 9 and 19–29 show different boards. The “border” of the board is represented by a yellow thick line, unless part of it or all of it is the edge of a face.

The board is tiled by a finite number of polygons. Construct polygonal plates in the same number, shape and size as the polygons of the board. Adjacent to each side of each plate draw a number, or two numbers, like it is shown in Figures 1 and 18–29. Figure 1 shows the obvious possibility of having plates with simple drawings, coloured drawings, etc.

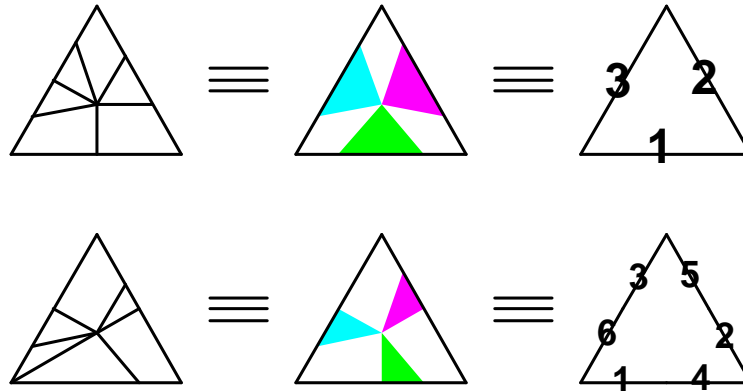


FIGURE 1.

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Now the game is to put the plates over the board polygons in such a way that the numbers near each board edge are equal. If there is at least one solution of this puzzle one says that we have a periodical plane puzzle with numbers.

These puzzles are a tool in teaching and learning mathematics. For those that already have some mathematical knowledge, they are a source for many examples and exercises, that go from the elementary to complex ones, in combinatorics, group theory (including symmetry and permutation groups), programming, and so on. The object of this work is to point out some possibilities by giving simple examples.

The computer is the only practical way of “materializing” infinite periodical plane puzzles. Hence, these puzzles can be very well put into practice as computer games.

This article follows some others on *puzzles with numbers*. See [3], [4], [5].

2. PLANE SYMMETRIES

A function $\omega : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is an *isometry* if it preserves the Euclidean distance: $\|\omega(\alpha) - \omega(\beta)\| = \|\alpha - \beta\|$, for every $\alpha, \beta \in \mathbb{R}^2$. We denote \mathcal{I} the group of isometries of \mathbb{R}^2 .

Every *translation* $\omega : \alpha \mapsto u + \alpha$, for some $u \in \mathbb{R}^2$, is an isometry.

Every linear isometry belongs to O_2 , the *orthogonal group*. We denote the identity by i .

Every isometry is a composition of a translation with an orthogonal transformation, i.e., it is of the form $\alpha \mapsto u + \eta(\alpha)$, for some $u \in \mathbb{R}^2$ and $\eta \in O_2$. The pair (u, η) defines the isometry. We represent the isometry by $\omega \equiv (u, \eta) \in \mathcal{I}$.

A *rotation* about the point γ is an isometry ω , $\omega(\alpha) = u + \eta(\alpha)$, if $\det \eta = 1$ and $\omega(\gamma) = \gamma$, i.e., $u = \gamma - \eta(\gamma)$. Hence, if $\eta \neq i$, $\gamma = (i - \eta)^{-1}(u)$. Notice that, if $\det \eta = 1$ and $\eta \neq i$, then $i - \eta$ is invertible.

If $\det \eta = 1$, one says that we have a *direct isometry*. Every direct isometry is a translation or a rotation[1]. The identity, $\omega = i$, is a rotation and a translation. If $\omega \neq i$ and $\eta = i$, then ω is a translation. If $\omega \neq i$ and $\eta \neq i$, then ω is a rotation.

If $\det \eta = -1$, every $\gamma \in \mathbb{R}^2$ can be decomposed $\gamma = \gamma_1 + \gamma_2$, with $\gamma_1 \perp \gamma_2$, such that $\eta(\gamma) = \eta(\gamma_1 + \gamma_2) = -\gamma_1 + \gamma_2$. The isometry $\omega(\alpha) = u + \eta(\alpha)$ is of the form $\omega(\alpha_1 + \alpha_2) = u_1 + u_2 - \alpha_1 + \alpha_2$. If $u_2 = 0$, ω is a *reflection*. If $u_2 \neq 0$, the isometry is called a *glide reflection*. The points in the “mirror” of reflection or glide reflection are $\alpha \in \mathbb{R}^2$ such that $\alpha_1 = u_1/2$.

If $\det \eta = -1$, one says that we have an *opposite isometry*. Every opposite isometry is a reflection or a glide reflection[1].

A *planer image* is a function $\xi : \mathbb{R}^2 \rightarrow D$, where D is a set, $D \neq \emptyset$.

From now on, let $\Omega_\xi \equiv \Omega$ be the group of isometries ω that leave ξ invariant:

$$\Omega_\xi \equiv \Omega = \{\omega \in \mathcal{I} : \xi \circ \omega = \xi\}.$$

In the following, Ω^+ denotes the subgroup of Ω of the isometries that preserve the orientation (direct isometries) and Ω^- denotes the subgroup of Ω of the isometries that reverse the orientation (opposite isometries).

We denote by Ω_* the group of orthogonal transformations associated with the isometries of Ω :

$$\Omega_* = \{\eta \in O_2 : (u, \eta) \in \Omega\}$$

In the following, if Λ is a finite set, then $|\Lambda|$ denotes its cardinal. Hence, if G is a finite group, $|G|$ denotes its order. For $a \in G$, $o(a)$ is the order of the group generated by a and $o(G) = \max \{o(a) : a \in G\}$.

2.1. The seventeen wallpaper groups. Let ξ be a planer image, as before. From now on, we assume that Ω is a discrete group of isometries invariant under two linearly independent translations which are of minimal length. Notice that Ω_* is finite. We say that ξ is a *pattern* and that Ω is a *wallpaper group*.

As it is very well known there are seventeen wallpaper groups. See [1], [6].

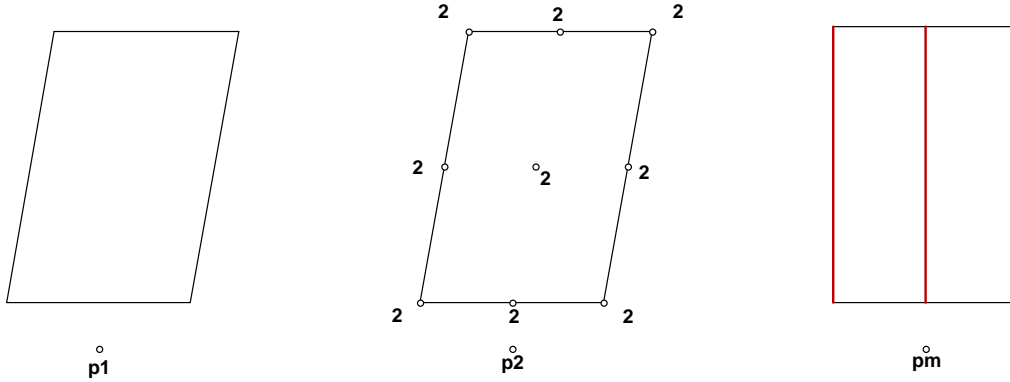


FIGURE 2.

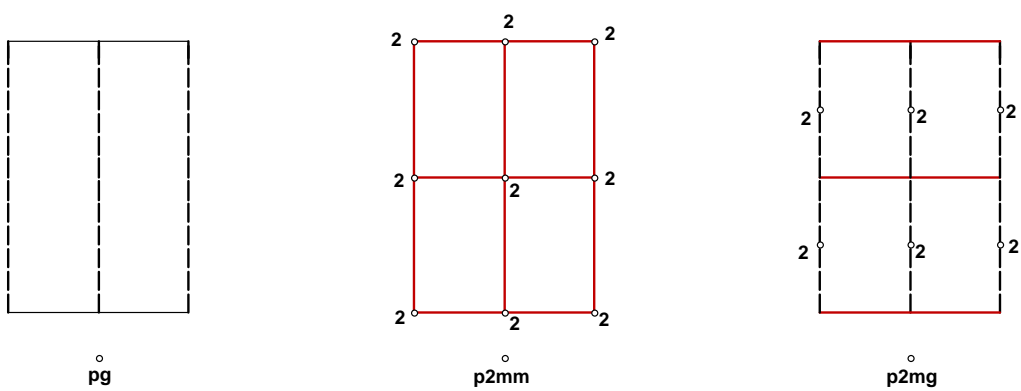


FIGURE 3.

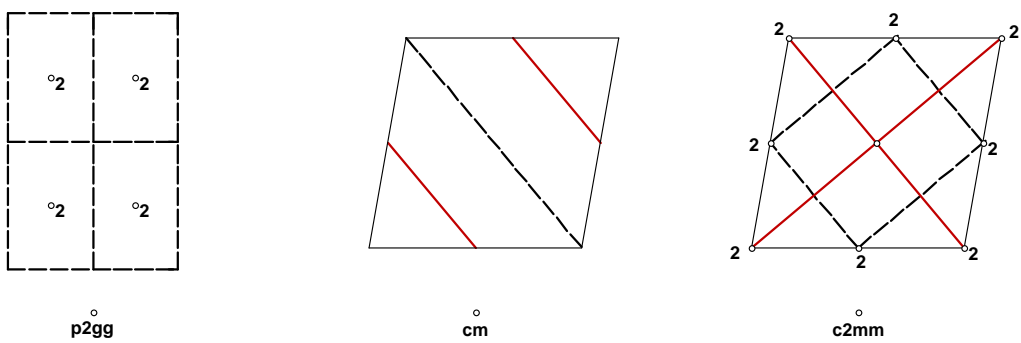


FIGURE 4.

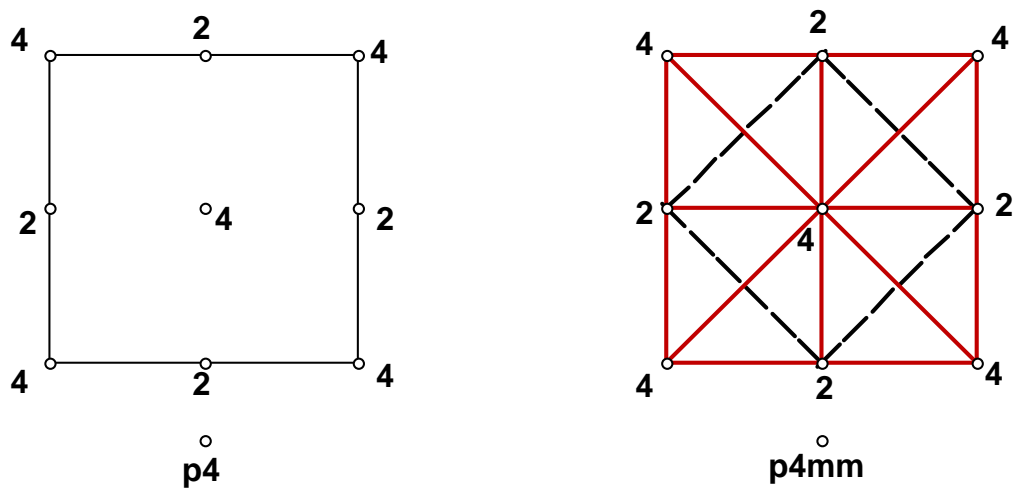


FIGURE 5.

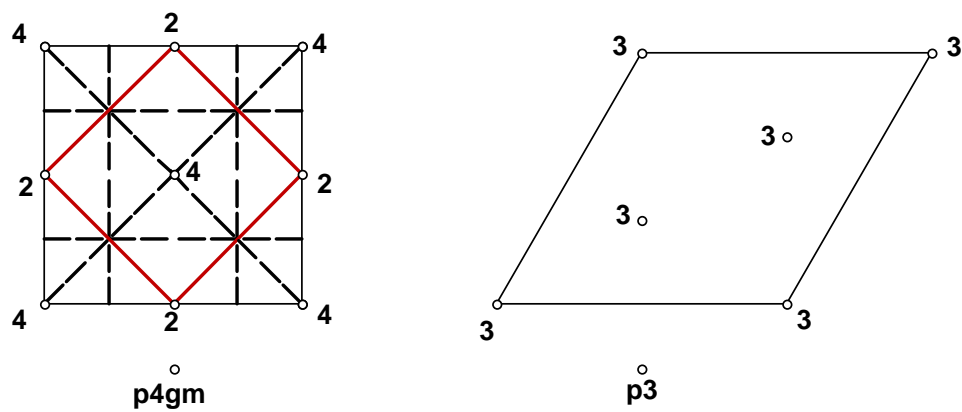


FIGURE 6.

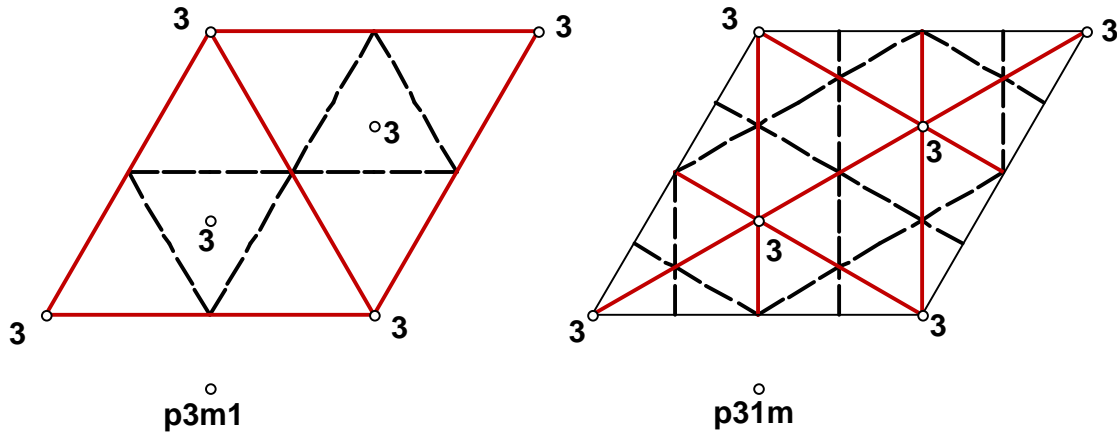


FIGURE 7.

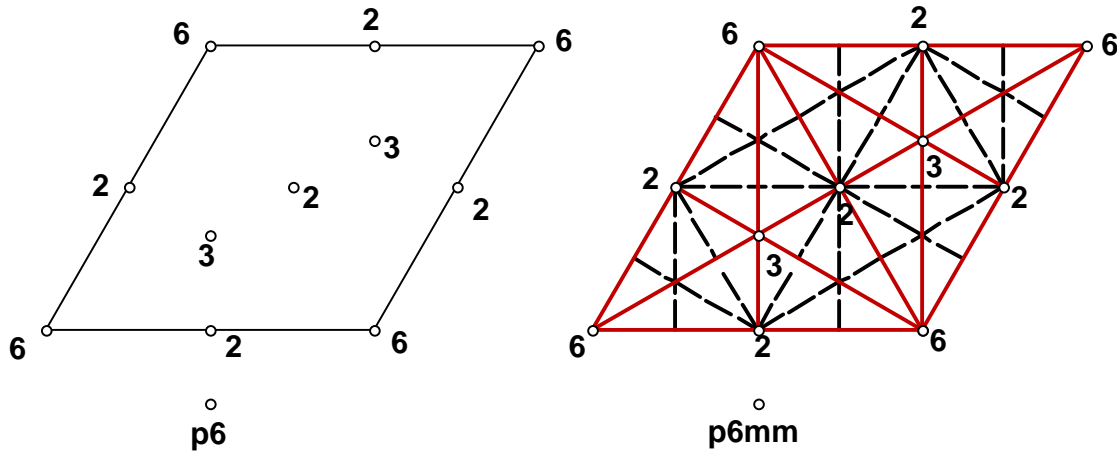


FIGURE 8.

Figures 2–8 represent fundamental regions of these seventeen groups. We use the notations of Ref. [1]. Here, small circles mean rotations and numbers their order. Mirrors are represented by thick red lines and glides by broken ones.

2.2. Wallpaper groups and permutation groups. In this article, S_n denotes the group of all permutations of $\{1, 2, \dots, n\}$; $a \in S_n$ means that a is a one-to-one function $a : \{1, 2, \dots, n\} \rightarrow \{1, 2, \dots, n\}$. The identity is $i : i(1) = 1, i(2) = 2, \dots, i(n) = n$. If $a_1, a_2 \in S_n$, we shall denote $a_1 a_2 \equiv a_1 \circ a_2$.

We shall write $a = (m_1 m_2 \cdots m_k) \cdots (n_1 n_2 \cdots n_l)$, if

$$\begin{aligned} a(m_1) &= m_2, a(m_2) = m_3, \dots, a(m_k) = m_1, \dots, \\ a(n_1) &= n_2, a(n_2) = n_3, \dots, a(n_l) = n_1 \end{aligned}$$

where $m_1, m_2, \dots, m_k, \dots, n_1, n_2, \dots, n_l \in \{1, 2, \dots, n\}$.

If $p \in \{1, 2, \dots, n\} \setminus \{m_1, m_2, \dots, m_k, \dots, n_1, n_2, \dots, n_l\}$, then $a(p) = p$.

The permutation $(m_1 m_2 \cdots m_k)$ is called a cyclic permutation, or a cycle (in this case a k -cycle); k is the length of the cyclic permutation.

Let $S_n^\pm = \{-1, 1\} \times S_n$. If $(\delta_1, a_1), (\delta_2, a_2) \in S_n^\pm$, then $(\delta_1, a_1)(\delta_2, a_2) = (\delta_1 \delta_2, a_1 a_2)$. S_n^\pm is a group. We shall note $(1, a) \equiv a$, $(-1, a) \equiv a_-$.

On permutation groups, see [8].

From now on, let Ω be a wallpaper group and let $\zeta : \Omega \rightarrow S_n^\pm$ be an homomorphism such that, if $\zeta(\omega) = (\delta, a)$, then $\delta = 1$ if ω preserves the orientation and $\delta = -1$ otherwise (a reflection or a glide reflection).

We shall say that the pair (Ω, ζ) is a *wallpaper group with permutations*. See [2], [7].

In the following two sections we impose that (Ω, ζ) is connected, i.e., if $n_1, n_2 \leq n$, then there exists $\omega \in \Omega$ such that $\zeta(\omega)(n_1) = n_2$.

3. TRANSLATIONS

Consider two independent vectors of the plane \mathbb{R}^2 , u and v , and Ω the group of translations generated by them: $\Omega = \{pu + qv : p, q \in \mathbb{Z}\}$. If $\zeta : \Omega \rightarrow S_n$ is a group homomorphism, then $\zeta(u)$ and $\zeta(v)$ commute and, of course, $\zeta(pu + qv) = \zeta(u)^p \zeta(v)^q$. We assume that if $\zeta(\Omega) \subset S_m$, then $m \geq n$.

One can identify Ω with \mathbb{Z}^2 by $(pu + qv) \leftrightarrow (p, q)$ and define

$$\tilde{\zeta}(p, q) = \zeta(pu + qv)(1).$$

Notice that ζ is connected if and only if $\tilde{\zeta}(\mathbb{Z}^2) = \{1, 2, \dots, n\}$.

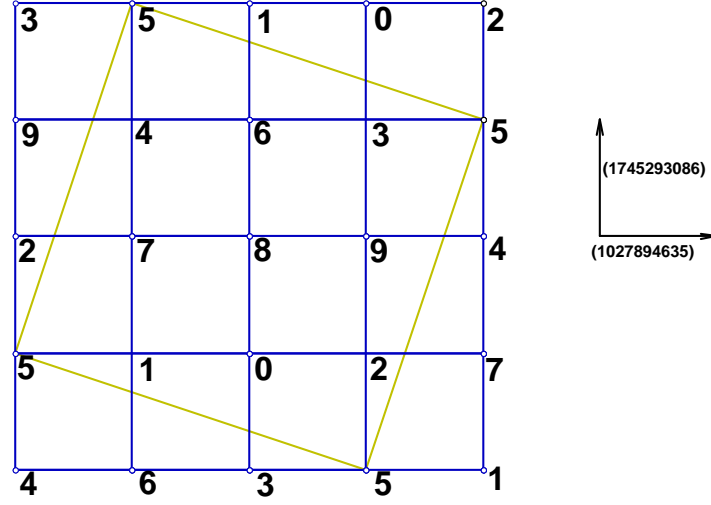


FIGURE 9.

As $\mathbb{Z}^2 \subset \mathbb{R}^2$, $\tilde{\zeta}$ generates a periodical pattern in the plane. A fundamental region of this pattern is a parallelogram and it contains exactly n points, (see Figures 9, 10 (a), 19 (a), 20 (a), 22 (a)). The vertices of this parallelogram in Figure 10 are $(0, 0)$, (p_1, q_1) , (p_2, q_2) and $(p_1 + p_2, q_1 + q_2)$. Hence

$$n = |p_1 q_2 - p_2 q_1|$$

The number of the $\zeta(u)$ cycles (μ_q) and the order of the $\zeta(u)$ cycles are

$$\mu_q = \gcd(|q_1|, |q_2|), \frac{n}{\mu_q},$$

and the number of the $\zeta(v)$ cycles (μ_p) and the order of the $\zeta(v)$ cycles are

$$\mu_p = \gcd(|p_1|, |p_2|), \frac{n}{\mu_p}.$$

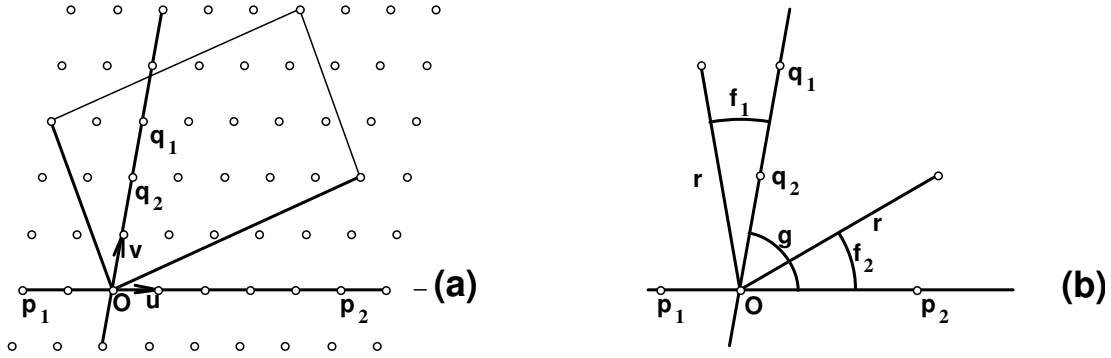


FIGURE 10.

Assume now that $\|u\| = \|v\|$ and that the parallelogram is equilateral (see Figure 10 (b)). Then

$$\frac{r}{\sin g} = \frac{-p_1}{\sin f_1} = \frac{q_1}{\sin (g + f_1)} = \frac{p_2}{\sin (g - f_2)} = \frac{q_2}{\sin f_2}.$$

1) If $f_1 = f_2$, then

$$\begin{aligned} q_2 &= -p_1, \quad p_2 = q_1 + 2p_1 \cos g, \\ n &= p_1^2 + q_1^2 + 2p_1 q_1 \cos g \\ &= p_2^2 + q_2^2 + 2p_2 q_2 \cos g. \end{aligned}$$

a) For $f_1 = f_2$, and $g = \frac{\pi}{3}$ (a regular triangular grid), then

$$q_2 = -p_1, \quad p_2 = p_1 + q_1, \quad n = p_1^2 + q_1^2 + p_1 q_1 = p_2^2 + q_2^2 + p_2 q_2.$$

The next table presents values of $n = p^2 + q^2 + pq$.

$p \rightarrow$ $q \downarrow$	0	1	2	3	4	5	6	...
0	0	1	4	9	16	25	36	...
1	1	3	7	13	21	31	43	...
2	4	7	12	19	28	39	52	...
3	9	13	19	27	37	49	63	...
4	16	21	28	37	48	61	76	...
...

Notice that the number of equilateral triangles in the fundamental region is $2n$.

b) For $f_1 = f_2$, and $g = \frac{\pi}{4}$ (a square grid), then

$$q_2 = -p_1, \quad p_2 = q_1, \quad n = p_1^2 + q_1^2 = p_2^2 + q_2^2.$$

The next table presents values of $n = p^2 + q^2$.

$p \rightarrow$ $q \downarrow$	0	1	2	3	4	5	6	...
0	0	1	4	9	16	25	36	...
1	1	2	5	10	17	26	37	...
2	4	5	8	13	20	29	40	...
3	9	10	13	18	25	34	45	...
4	16	17	20	25	32	41	52	...
...

2) If $f_1 = -f_2$, then

$$q_2 = p_1, \quad p_2 = q_1, \quad n = |p_1^2 - q_1^2| = |p_2^2 - q_2^2|.$$

Examples:

	p_1	q_1	p_2	q_2	n	μ_q	$\frac{n}{\mu_q}$	μ_p	$\frac{n}{\mu_p}$
Figure 9	-3	1	1	3	10	1	10	1	10
Figure 10 (a)	-2	3	5	2	19	1	19	1	19
Figure 19 (a)	-1	2	2	1	5	1	5	1	5
Figure 20 (a)	-2	2	2	2	8	2	4	2	4
Figure 22 (a)	-3	1	-2	3	7	1	7	1	7

4. ROTATIONS AND REFLECTIONS

In this section we describe the different possibilities for wallpaper groups with permutations. Here, the translations are generated by rotations, reflections and glide reflections. We impose that (Ω, ζ) is connected and that $n \leq o(\Omega_*)$. As before, in the figures, small circles mean rotations and numbers their order. Mirrors are represented by thick red lines and glides by broken ones. From now on, the letters $a, b, c, d, x, y, z, w, u$ and v represent permutations.

Let us describe, briefly, the method we follow in this section.

If ω_1 is a rotation of order k , and ω_2 is a rotation of order j , then ω_1 transforms ω_2 in another rotation of order j , ω_3 , which is $\omega_1 \omega_2 \omega_1^{-1} \equiv \omega_1(\omega_2)$.

If one wants to translate the isometries into elements of S_n^\pm the function must be such that if $\omega_1 \mapsto s_1$, $\omega_2 \mapsto s_2$, then $\omega_3 \equiv \omega_1(\omega_2) \mapsto s_1 s_2 s_1^{-1}$.

Note that if $s_1 = (\delta_1, a_1)$, $s_2 = (\delta_2, a_2)$, where

$$a_2 = (m_1 m_2 \cdots m_l) \cdots,$$

then $\omega_3 \equiv \omega_1 (\omega_2) \mapsto s_3 = (\delta_2, a_3)$, with

$$a_3 = (a_1 (m_1) a_1 (m_2) \cdots a_1 (m_l)) \cdots.$$

We assign to every axis of order $k (\equiv \omega)$, a permutation a of order k_1 , a divisor of k , so that to the counter clock-wise rotation of $\frac{2\pi}{k}$, ω , corresponds the permutation a . This association must be coherent in the sense that it generates a group homomorphism.

If we assign to ω and ω_1 (two neighbor axes) the permutations a and a_1 , then to the axis $\omega (\omega_1) = \omega \omega_1 \omega^{-1}$ we must assign aa_1a^{-1} . When $a_1 = (m_1 m_2 \cdots m_l) \cdots$, then $aa_1a^{-1} = (a(m_1) a(m_2) \cdots a(m_l)) \cdots$.

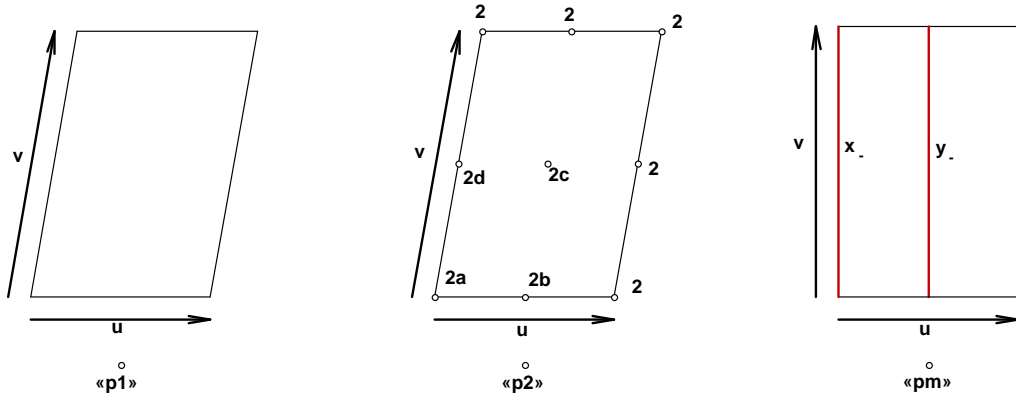


FIGURE 11.

4.1. **Figure 11, «p2».** In the case «p2», the possibilities for a , b , c and d are i or (12) . Notice that $c = dba$, $u = ba$ and $v = da$.

a	b	d	c	u	v
i	i	i	i	i	i
(12)	i	i	(12)	(12)	(12)
(12)	i	(12)	i	(12)	i
(12)	(12)	(12)	(12)	i	i

4.2. **Figure 11, «pm».** In the case «pm», the possibilities for x and y are i or (12) . Notice that $u = xy$ and $v = i$.

x	y	u	v
i	i	i	i
(12)	i	(12)	i
(12)	(12)	i	i

4.3. **Figure 12, «pg».** In the case «pg», the possibilities for x and y are i or (12) . Notice that $u = xy$ and $v = i$.

x	y	u	v
i	i	i	i
(12)	i	(12)	i
(12)	(12)	i	i

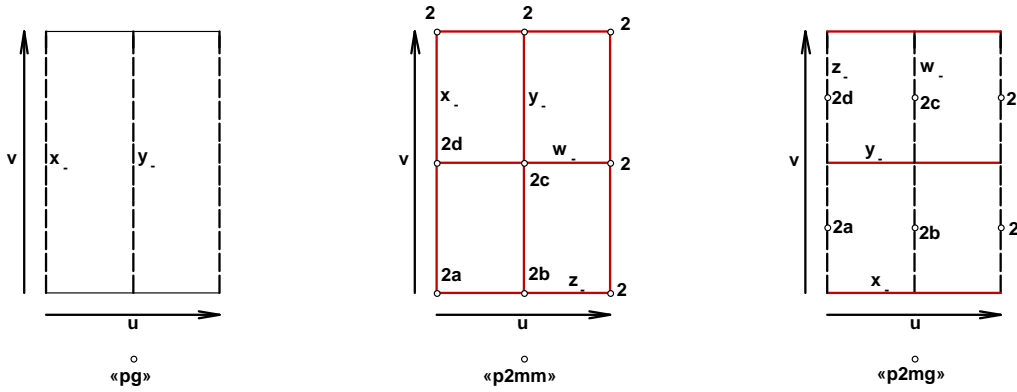


FIGURE 12.

4.4. **Figure 12, «p2mm».** In the case «p2mm», the possibilities for a, b, c and d are i or (12) as in «p2». The possibilities for x and y are i or (12) .

Notice that $c = dba$, $u = ba$, $v = da$, $y = ux$, $z = ax$, and $w = dx$.

a	b	d	c	u	v	x	y	z	w
i	i	i	i	i	i	i	i	i	i
i	i	i	i	i	i	(12)	(12)	(12)	(12)
(12)	i	i	(12)	(12)	(12)	i	(12)	(12)	i
(12)	i	(12)	i	(12)	i	i	(12)	(12)	(12)
(12)	i	(12)	i	(12)	i	(12)	i	i	i
(12)	(12)	(12)	(12)	i	i	i	i	(12)	(12)

4.5. **Figure 12, «p2mg».** In the case «p2mg», the possibilities for a, b, c and d are i or (12) as in «p2». The possibilities for x and y are i or (12) .

Notice that $d = a, c = b, u = ba, v = i, y = x, z = ax, w = bx$.

a	b	d	c	u	v	x, y	z	w
i	i	i	i	i	i	i	i	i
i	i	i	i	i	i	(12)	(12)	(12)
(12)	i	(12)	i	(12)	i	i	(12)	i
(12)	i	(12)	i	(12)	i	(12)	i	(12)
(12)	(12)	(12)	(12)	i	i	i	(12)	(12)
(12)	(12)	(12)	(12)	i	i	(12)	i	i

4.6. **Figure 13, «p2gg».** In the case «p2gg», the possibilities for a, b, c and d are i or (12) as in «p2». The possibilities for x and y are i or (12).

Notice that $a = b = c = d, u = i, v = i, y = x, z = w = ax$.

a	b	d	c	u	v	x, y	z, w
i	i	i	i	i	i	i	i
i	i	i	i	i	i	(12)	(12)
(12)	(12)	(12)	(12)	i	i	i	(12)

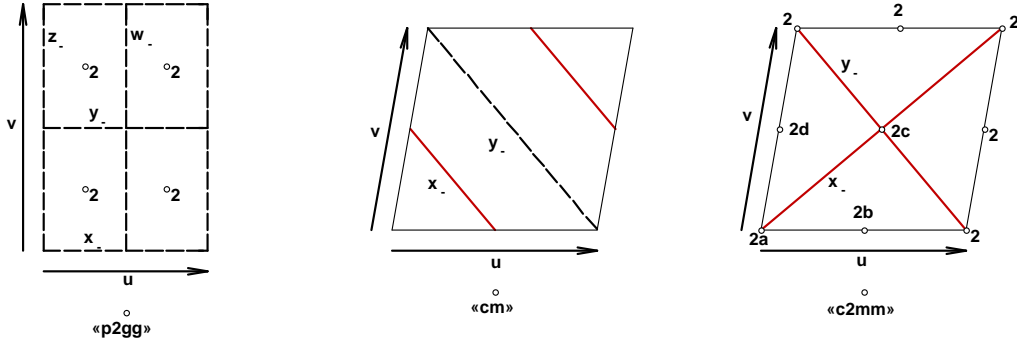


FIGURE 13.

4.7. **Figure 13, «cm».** In the case «cm», the possibilities for x and y are i or (12).

Notice that $u = v = xy$.

x	y	u	v
i	i	i	i
i	(12)	(12)	(12)
(12)	i	(12)	(12)
(12)	(12)	i	i

4.8. **Figure 13, «c2mm».** In the case «c2mm», the possibilities for a, b, c and d are i or (12) as in «p2». The possibilities for x and y are i or (12) .

Notice that $b = d, c = a, u = v = ba, v = da, y = ax$.

a	b	d	c	u, v	x	y
i	i	i	i	i	i	i
i	i	i	i	i	(12)	(12)
i	(12)	(12)	i	(12)	i	i
i	(12)	(12)	i	(12)	(12)	i
(12)	i	i	(12)	(12)	i	(12)
(12)	(12)	(12)	(12)	i	i	(12)

4.9. **Figure 14, «p4».** In the case «p4», $b = ca, d = ac, u = ca^{-1}, v = c^{-1}a$. We shall take always a and c such that $o(c) \leq o(a)$.

The possibilities for a are i (for $n = 1$) and permutations of the type (12) (for $n = 2$), $(12)(34)$ and (1234) (for $n = 4$).

If $a = i$, then $c = i$. If $a = (12)$, then $c = i$ or $c = (12)$.

If $a = (12)(34)$, then $c = (13)(24)$.

If $a = (1234)$, then $cac^{-1} = (1234)$ or (1432) , hence the possibilities for c are (1234) , (1432) , (13) , (24) , $(12)(34)$ and $(14)(23)$. Notice that, since $a(13)a^{-1} = (24)$ and $a(12)(34)a^{-1} = (14)(23)$, the possibilities for c are, in fact, (1234) , (1432) , (13) and $(12)(34)$.

a	c	b	u	v
i	i	i	i	i
(12)	i	(12)	(12)	(12)
(12)	(12)	i	i	i
$(12)(34)$	$(13)(24)$	$(14)(23)$	$(14)(23)$	$(14)(23)$
(1234)	(1234)	$(13)(24)$	i	i
(1234)	(1432)	i	$(13)(24)$	$(13)(24)$
(1234)	(13)	$(12)(34)$	$(14)(23)$	$(12)(34)$
(1234)	$(12)(34)$	(24)	(13)	(24)

4.10. **Figure 14, «p4mm».** In the case «p4mm», a, b, c, d, u and v are as in «p4».

Here, $y = a^{-1}x$.

For $a = i, n \leq 2$, the possibilities for x are i and (12) .

For $a = (12), n = 2$, the possibilities for x are i and (12) .

For $a = (12)(34), c = (13)(24)$, the possibilities for x are $i, (12)(34), (13)(24)$ and $(14)(23)$.

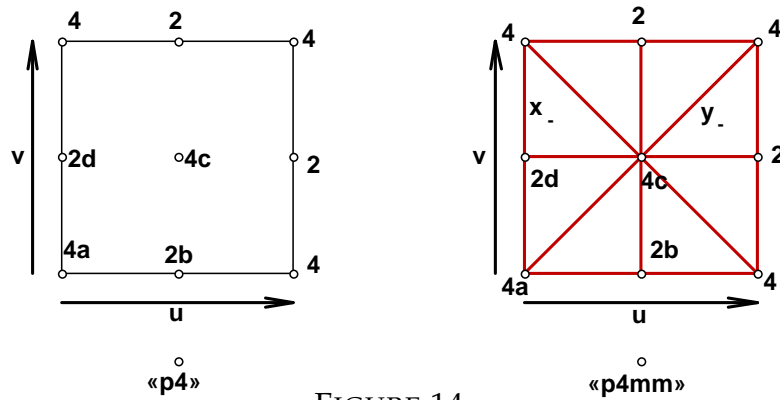


FIGURE 14.

For $a = (1234)$, $c = (1234)$, the possibilities for x are (13) and $(12)(34)$.

For $a = (1234)$, $c = (1432)$, the possibilities for x are (13) and $(12)(34)$.

For $a = (1234)$, $c = (13)$, the possibilities for x are $(12)(34)$ and $(14)(23)$.

For $a = (1234)$, $c = (12)(34)$, the possibilities for x are (13) and (24) .

As «p4mm» is constructed from «p4» adding reflections, these reflections can connect permutations:

a) $a = (12)(34)$, $c = i$, $b = (12)(34)$, $u = (12)(34)$, $x = (13)(24)$.

b) $a = (12)(34)$, $c = (12)(34)$, $b = i$, $u = i$, $x = (13)(24)$.

a	c	b	u	x	y
i	i	i	i	i	i
i	i	i	i	(12)	(12)
(12)	i	(12)	(12)	i	(12)
(12)	i	(12)	(12)	(12)	i
(12) (34)	i	(12) (34)	(12) (34)	(13) (24)	(14) (23)
(12)	(12)	i	i	i	(12)
(12)	(12)	i	i	(12)	i
(12) (34)	(12) (34)	i	i	(13) (24)	(14) (23)
(12) (34)	(13) (24)	(14) (23)	(14) (23)	i	(12) (34)
(12) (34)	(13) (24)	(14) (23)	(14) (23)	(12) (34)	i
(12) (34)	(13) (24)	(14) (23)	(14) (23)	(13) (24)	(14) (23)
(12) (34)	(13) (24)	(14) (23)	(14) (23)	(14) (23)	(13) (24)
(1234)	(1234)	(13) (24)	i	(13)	(12) (34)
(1234)	(1234)	(13) (24)	i	(12) (34)	(24)
(1234)	(1432)	i	(13) (24)	(13)	(12) (34)
(1234)	(1432)	i	(13) (24)	(12) (34)	(24)
(1234)	(13)	(12) (34)	(14) (23)	(12) (34)	(24)
(1234)	(13)	(12) (34)	(14) (23)	(14) (23)	(13)
(1234)	(12) (34)	(24)	(13)	(24)	(14) (23)
(1234)	(12) (34)	(24)	(13)	(13)	(12) (34)

4.11. **Figure 15, « $p4gm$ ».** In the case « $p4gm$ », a, b, c, d, u and v are as in « $p4$ ».

Here, $y = cxc^{-1}$.

For $a = i, n \leq 2$, the possibilities for x are i and (12).

For $a = (12), n = 2$, the possibilities for x are i and (12).

For $a = (12) (34), c = (13) (24)$, the possibilities for x are (14) and (23).

For $a = (1234), c = (1234)$, the possibilities for x are (13), (24), (12) (34) and (14) (23).

For $a = (1234), c = (1432)$, the possibilities for x are i and (13) (24).

As « $p4gm$ » is constructed from « $p4$ » adding reflections, these reflections can connect permutations:

a) $a = (12), c = (34), b = (12) (34), u = i, x = (13) (24)$.

b) $a = (12) (34), c = (12) (34), b = i, u = i, x = (13) (24)$.

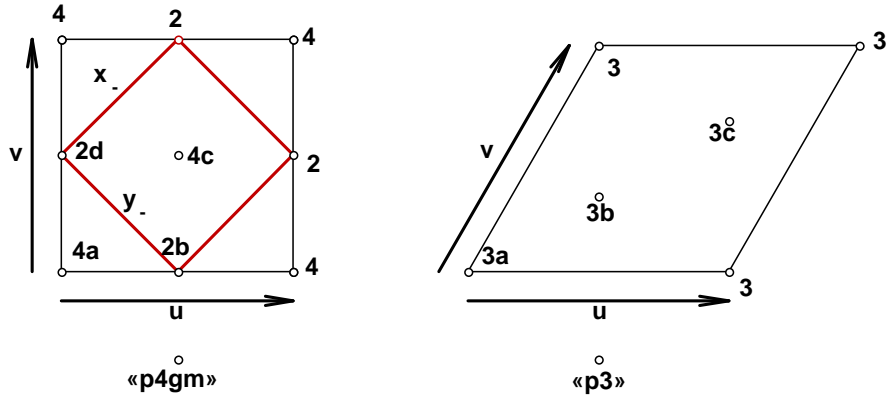


FIGURE 15.

a	c	b	u	x	y
i	i	i	i	i	i
i	i	i	i	(12)	(12)
(12)	(34)	(12) (34)	i	(13) (24)	(14) (23)
(12)	(12)	i	i	i	i
(12)	(12)	i	i	(12)	(12)
(12) (34)	(12) (34)	i	i	(13) (24)	(13) (24)
(12) (34)	(13) (24)	(14) (23)	(14) (23)	(14)	(23)
(1234)	(1234)	(13) (24)	i	(13)	(24)
(1234)	(1234)	(13) (24)	i	(12) (34)	(14) (23)
(1234)	(1432)	i	(13) (24)	i	i
(1234)	(1432)	i	(13) (24)	(13) (24)	(13) (24)

4.12. **Figure 15, «p3».** In the case «p3», $c = ba^2b$, $u = ba^2$, $v = b^2a$. We shall take always a and b such that $o(a) \leq o(b)$, $o(c)$.

The possibilities for a are i (for $n = 1$) and permutations of the type (123) (for $n = 3$).

For $a = i$, the possibilities for b are i (for $n = 1$) and permutations of the type (123) (for $n = 3$).

For $a = (123)$, the possibility for b is (123).

a	b	c	u	v
i	i	i	i	i
i	(123)	(132)	(123)	(132)
(123)	(123)	(123)	i	i

4.13. **Figure 16, «p3m1».** For $a = b = i$, the possibilities for x are i and permutations of the type (12).

For $a = (123)$ or $b = (123)$, the possibilities for x are (12), (23) and (13).

a	b	c	u	v	x
i	i	i	i	i	i
i	i	i	i	i	(12)
i	(123)	(132)	(123)	(132)	i
(123)	(123)	(123)	i	i	(12)

4.14. **Figure 16, «p31m».** For $a = b = i$, the possibilities for x are i and permutations of the type (12).

For $a = (123)$ or $b = (123)$, the possibilities for x are (12), (23) and (13).

a	b	c	u	v	x
i	i	i	i	i	i
i	i	i	i	i	(12)
i	(123)	(132)	(123)	(132)	(12)
(123)	(123)	(123)	i	i	(12)

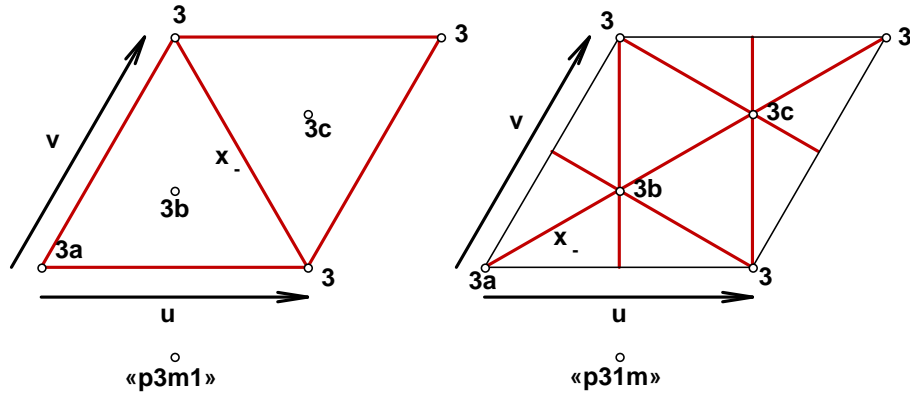


FIGURE 16.

4.15. **Figure 17, «p6».** In the case «p6», $c = ad$, $b = ca$, $u = ca^{-2}$, $v = c^{-1}a^2$. The permutations a and d are compatible if and only if they obey the rule $ada = dad$.

For $n = 1$, $a = d = i$.

If $o(a) = o(d) = 2$, the possibilities are the following:

- a) For $n = 2$, $a = d = (12)$.
- b) For $n = 3$, $a = (12)$, $d = (13)$.
- c) For $n = 6$, $a = (12)(34)(56)$, $d = (13)(25)(46)$.

If $o(a) = o(d) = 3$, the possibilities are the following:

- a) For $n = 3$, $a = d = (123)$.
- b) For $n = 4$, $a = (123)$, $d = (142)$.
- c) For $n = 6$, $a = (123)(456)$, $d = (124)(356)$.

If $o(a) = o(d) = 6$, the possibilities are the following:

- a) $a = d = (123456)$.
- b) $a = (123456)$, $d = (156423)$. There are other two possibilities for d but they are generated by the action of a on *this* d .
- c) $a = (123456)$, $d = (163254)$. There is other possibility for d but it is generated by the action of a on *this* d .

a	d	c	b	u
i	i	i	i	i
(12)	(12)	i	(12)	i
(12)	(13)	(132)	(23)	(132)
(12)(34)(56)	(13)(25)(46)	(145)(263)	(16)(24)(35)	(145)(263)
(123)	(123)	(132)	i	i
(123)	(142)	(143)	(12)(34)	(12)(34)
(123)(456)	(124)(356)	(136)(254)	(15)(26)	(15)(26)
(123456)	(123456)	(135)(246)	(14)(25)(36)	i
(123456)	(156423)	(165)(243)	(14)	(25)(36)
(123456)	(163254)	(264)	(16)(23)(45)	(153)(246)

4.16. **Figure 17, «p6mm».** In the case «p6mm», a, b, c, d, u and v are as in «p6» and $y = xa^{-1}$.

As «p6mm» is constructed from «p6» adding reflections, these reflections can connect permutations:

- a) $a = d = (12)(34)$. In this case $c = i$, $b = (12)(34)$, $u = v = i$.
- b) $a = (12)(45)$, $d = (13)(46)$. In this case $c = (132)(465)$, $b = (23)(56)$, $u = (132)(465)$, $v = (123)(456)$.

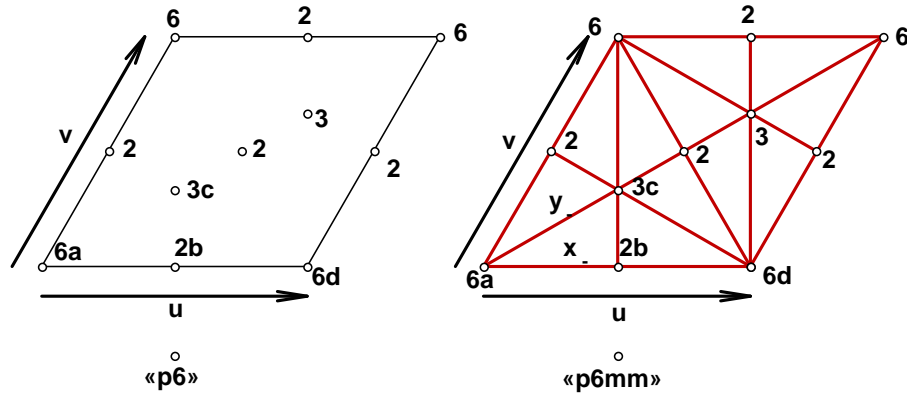


FIGURE 17.

c) $a = d = (123)(456)$. In this case $c = (132)(465)$, $b = i$, $u = v = i$.

a	d	x	y
i	i	i	i
i	i	(12)	(12)
(12)	(12)	i	(12)
(12)	(12)	(12)	i
$(12)(34)$	$(12)(34)$	$(13)(24)$	$(14)(23)$
(12)	(13)	i	(12)
$(12)(45)$	$(13)(46)$	$(14)(25)(36)$	$(15)(24)(36)$
$(12)(34)(56)$	$(13)(25)(46)$	i	$(12)(34)(56)$
$(12)(34)(56)$	$(13)(25)(46)$	$(12)(35)(46)$	$(36)(45)$
$(12)(34)(56)$	$(13)(25)(46)$	$(16)(25)(34)$	$(15)(26)$
$(12)(34)(56)$	$(13)(25)(46)$	$(13)(24)(56)$	$(14)(23)$
(123)	(123)	(12)	(13)
$(123)(456)$	$(123)(456)$	$(14)(26)(35)$	$(15)(24)(36)$
(123)	(142)	(12)	(13)
$(123)(456)$	$(124)(356)$	$(12)(56)$	$(13)(45)$
$(123)(456)$	$(124)(356)$	$(16)(25)(34)$	$(14)(26)(35)$
(123456)	(123456)	$(14)(23)(56)$	$(15)(24)$
(123456)	(123456)	$(26)(35)$	$(12)(36)(45)$
(123456)	(156423)	$(14)(23)(56)$	$(15)(24)$
(123456)	(156423)	$(26)(35)$	$(12)(36)(45)$
(123456)	(163254)	$(14)(23)(56)$	$(15)(24)$

5. PLANE SYMMETRIES, PERMUTATION GROUPS AND PUZZLE SOLUTIONS

5.1. Board symmetries. Consider a board with at least three faces with a common vertex. From now on V denotes the set of the board vertices, E denotes the set of the board edges and F denotes the set of the board faces (polygons).

The group of the board symmetries, Ω , called the board group, is the set of all isometries ω of \mathbb{R}^2 , $\omega \equiv (u, \eta)$, that send vertices to vertices, which implies that they send edges to edges, faces to faces. Every symmetry $\omega \in \Omega$ induces three bijections, that we shall also denote ω , whenever there is no confusion possible: $\omega : V \rightarrow V$, $\omega : E \rightarrow E$ and $\omega : F \rightarrow F$. Denote also $\Omega \equiv \{\omega : V \rightarrow V\} \equiv \{\omega : E \rightarrow E\} \equiv \{\omega : F \rightarrow F\}$, the three sets of these functions. One can say that each one of these three sets Ω is the set of the board symmetries. Notice that not all one-to-one functions $F \rightarrow F$, $E \rightarrow E$, $V \rightarrow V$ are in Ω . With the composition of functions each one of these three sets Ω forms a group that is isomorphic to the group of the board symmetries. If $\omega_1, \omega_2 \in \Omega$, we shall denote $\omega_1 \omega_2 \equiv \omega_1 \circ \omega_2$.

When no confusion is possible, $\omega \in \Omega$ represents also the group isomorphism $\omega : \Omega \rightarrow \Omega$, $\omega(\omega_1) = \omega \omega_1 \omega^{-1}$, for every $\omega_1 \in \Omega$. Note that ω_1 and $\omega(\omega_1)$ have the same order.

If Ω_1 is a subgroup of Ω , then Ω_1 acts naturally on the face set, F : for $\omega \in \Omega_1$ and $\varphi \in F$, one defines the action $\omega \varphi = \omega(\varphi)$.

5.2. Puzzle solutions. Consider a puzzle with numbers $1, 2, \dots, n$ drawn on the plates. From now on P denotes the set of its plates which have numbers drawn, and call it the plate set. If no confusion is possible, P will also denote the puzzle itself. However, note that we can not separate the plates from the board: the puzzle is the plates *and* the board.

As before E denotes the set of the board edges and F denotes the set of the board faces. A solution of the puzzle defines a function $\varepsilon : E \rightarrow \{1, 2, \dots, n\}$. Denote \mathcal{E} the set of these functions. One can say that \mathcal{E} is the set of the puzzle solutions.

We shall also consider the group $S_n \times \Omega$. If $(a_1, \omega_1), (a_2, \omega_2) \in S_n \times \Omega$, one defines the product $(a_1, \omega_1)(a_2, \omega_2) = (a_1 a_2, \omega_1 \omega_2)$.

5.3. The plate group. Some S_n subgroups act naturally on P . Let $\pi \in P$ and $a \in S_n$. Assume that m_1, m_2, m_3, \dots are drawn on π , by this order. Then $a\pi$ is a plate where the numbers $a(m_1) = n_1, a(m_2) = n_2, a(m_3) = n_3, \dots$ are drawn replacing m_1, m_2, m_3, \dots (see Figure 18).

Let $s \in S_n^\pm$ and $\pi \in P$. If $s \equiv s_1 = (1, a) \equiv a$, then $s\pi = a\pi$. If $s \equiv s_2 = (-1, a) \equiv a^-$, then $s\pi$ is a reflection of $a\pi$. In this last case, if the numbers m_1, m_2, m_3, \dots

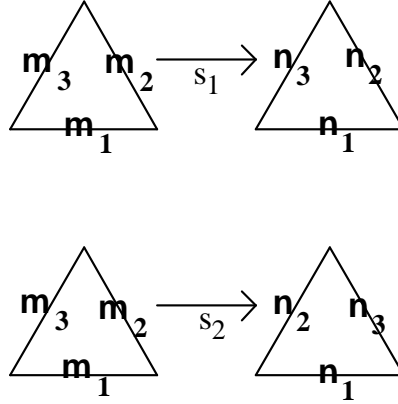


FIGURE 18.

are drawn on π , by this order, then $s\pi$ is a plate where the numbers $\dots, a(m_3) = n_3, a(m_2) = n_2, a(m_1) = n_1$ are drawn by this order (see Figure 18).

The plate group, G_P , is the greatest subgroup of S_n^\pm (if $\Omega^- \neq \emptyset$) that acts on P , or the greatest subgroup of S_n (if $\Omega^- = \emptyset$) that acts on P . For $\Omega^- \neq \emptyset$, if $s \in S_n^\pm$ and $s\pi \in P$, for every $\pi \in P$, then $s \in G_P$. For $\Omega^- = \emptyset$, if $s \in S_n$ and $s\pi \in P$, for every $\pi \in P$, then $s \in G_P$.

5.4. The solution group. Let $\varepsilon : E \rightarrow \{1, 2, \dots, n\}$ be a solution of the puzzle. The group of this solution, G_ε , is a subgroup of $S_n \times \Omega$; $(a, \omega) \in G_\varepsilon$ if and only if

$$a \circ \varepsilon = \varepsilon \circ \omega.$$

Denote Ω_ε the following subgroup of Ω : $\omega \in \Omega_\varepsilon$ if and only if there exists $a \in S_n$ such that $(a, \omega) \in G_\varepsilon$. Notice that if $\omega \in \Omega_\varepsilon$ there exists only one $a \in S_n$ such that $(a, \omega) \in G_\varepsilon$. From this one concludes that $\omega \mapsto (a, \omega)$ defines an isomorphism between Ω_ε and G_ε and that $(\det \eta, a) \in G_P$. This defines $g_\varepsilon : \Omega_\varepsilon \rightarrow G_P$, $g_\varepsilon(\omega) = (\det \eta, a)$, which is an homomorphism of groups.

For a lot of puzzles $(\det \eta, a)$ defines completely ω . It is the case of all puzzles considered in this article. Hence, when $(\det \eta, a)$ defines completely ω , g_ε establishes an isomorphism between Ω_ε and $g_\varepsilon(\Omega_\varepsilon) \subset G_P$. Denote $G_{P_\varepsilon} \equiv g_\varepsilon(\Omega_\varepsilon)$. Finally, G_ε and G_{P_ε} are isomorphic. We can identify (a, ω) with $(\det \eta, a)$, and G_ε with the subgroup G_{P_ε} of G_P .

5.5. Equivalent solutions. Let $\varepsilon_1, \varepsilon_2 : E \rightarrow \{1, 2, \dots, n\}$ be solutions of the puzzle. One says that these solutions are equivalent, $\varepsilon_1 \approx \varepsilon_2$, if there are $\omega \in \Omega$, $\omega \equiv (u, \eta)$, and $a \in S_n$, such that

$$a \circ \varepsilon_1 = \varepsilon_2 \circ \omega.$$

Notice that $(\det \eta, a) \in G_P$.

If $a = i$ and $\det \eta = 1$, what distinguishes the solutions ε_1 and ε_2 is only a symmetry in Ω^+ . In this case

$$\varepsilon_1 = \varepsilon_2 \circ \omega$$

expresses another equivalence relation, $\varepsilon_1 \sim \varepsilon_2$. When we make a puzzle, in practice, we do not recognize the difference between ε_1 and ε_2 . We shall say that they represent the same *natural solution*, an equivalence class of the relation \sim .

Let $\varepsilon, \varepsilon_1, \varepsilon_2 \in \mathcal{E}$. As $\varepsilon_1 \sim \varepsilon_2$ and $\varepsilon_1 \approx \varepsilon$, implies $\varepsilon_2 \approx \varepsilon$, one can say that the natural solution represented by ε_1 is equivalent to ε .

This equation involving ε_1 and ε_2 defines an equivalence relation, and a natural solution is an equivalence class of this relation. Notice that if $\varepsilon_1 = \varepsilon_2$, then ω_1 is the identity.

For $\varepsilon \in \mathcal{E}$, represent by $[\varepsilon]$ the set of natural solutions equivalent to ε .

Now, if $\Omega^- \neq \emptyset$, choose $\omega_- \in \Omega^-$. For $\varepsilon \in \mathcal{E}$ and $s = (\delta, a) \in G_P$, denote $\varepsilon_s = a \circ \varepsilon \circ \omega$, where ω is the identity if $\delta = 1$ and $\omega = \omega_-$ if $\delta = -1$. The set $\{\varepsilon_s : s \in G_P\}$ includes representatives of all natural solutions equivalent to ε . Then

$$|[\varepsilon]| = \frac{|G_P|}{|G_{P_\varepsilon}|}.$$

The cardinal of all the natural solutions is then given by

$$\sum_{[\varepsilon]} \frac{|G_P|}{|G_{P_\varepsilon}|},$$

where the sum is extended to all different equivalence classes $[\varepsilon]$.

5.6. Equivalent puzzles. Consider two puzzles and their plate sets, P_1 and P_2 . Let $\delta = 1$ if they have the same board, but $\Omega^- = \emptyset$; $\delta = -1$ if they have different boards, but symmetric by reflection; and $\delta = \pm 1$ if they have the same board and $\Omega^- \neq \emptyset$. One says that the puzzles are equivalent if there exists $s = (\delta, a) \in S_n^\pm$ such that the function $\pi \mapsto s\pi$ is one-to-one between P_1 and P_2 . We denote P_2 as sP_1 .

As an example take the puzzles in Figure 20 (c) and (d). Let $a = (13)(67)$. If $s = (-1, a)$ acts on the plates of one of these puzzles, it generates the plates of the other. If P is the plate set of one of these puzzles, then $P \neq sP$.

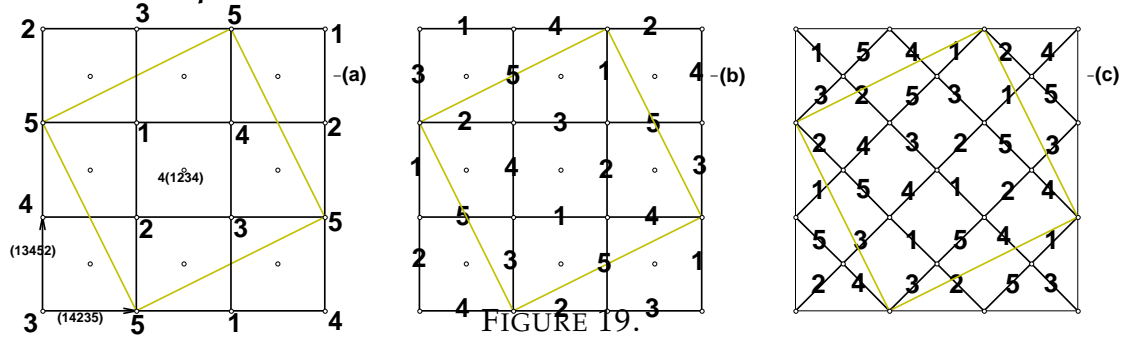
6. EXAMPLES

In this section we use group theory in order to find puzzles, for a given board, such as, for example, maximal puzzles. These are important examples, but others could be given.

To avoid ambiguities, in the puzzles we give in the following, all the edges have numbers.

Some of the examples we give in this section can easily be studied directly. All the results we present are obtained in this way. Some puzzles have only one natural solution. Others can have millions of natural solutions that can only be calculated with a computer.

6.1. Puzzles «p4».



6.1.1. *Figure 19 (a).* Take a board with five square faces (Figure 19 (a)). Obviously, it is not symmetric by reflection. The group of this board has 20 elements.

The group of permutations in S_5 associated with translations of this pattern are generated by $u = (14235)$ and $v = (13452)$. The group of this pattern is generated by

a	c	b	u	v
(1234)	(1325)	$(15)(34)$	(14235)	(13452)

6.1.2. *Figure 19 (b).* In this figure it is represented a maximal solution of a puzzle with plates $[1234]$, $[1325]$, $[1453]$, $[1542]$, $[2435]$. It is not very interesting because, in fact, this natural solution is unique. Its group has 20 elements.

6.1.3. *Figure 19 (c).* One can construct another board with ten square faces with vertices in the ten rotations centers of the grid (Figure 19 (c)). The group of this board has 40 elements.

In Figure 19 (c), is represented a maximal solution of a puzzle with plates $[1415]$, $[1213]$, $[2125]$, $[2324]$, $[3134]$, $[3235]$, $[4142]$, $[4345]$, $[5153]$, $[5254]$. Its group has 20 elements.

6.2. Puzzles « $p4mm$ ».

6.2.1. *Figure 20 (a)*. The board with eight faces represented in Figure 20 (a) is symmetric by reflection. The group of this board has 64 elements.

The group of permutations in S_8 associated with translations of the pattern in the point lattice is generated by $u = (1835)(2647)$ and $v = (1637)(2845)$. The group of this pattern is generated by

a	c	b
$(1234)(67)$	$(1265)(3748)$	$(15)(27)(38)(46)$

u	x	y
$(1835)(2647)$	$(13)(67)$	$(12)(34)$

There are no puzzles with solutions that are “compatible” with Ω^+ .

One can construct another board with sixteen square faces with vertices in the sixteen rotations centers of the grid. The group of this board has 128 elements.

There are three puzzles with solutions that are “compatible” with Ω^+ . In Figure 20 (b), (c) and (d) we represent their unique maximal solutions. Puzzles (c) and (d) are equivalent.

6.2.2. *Figure 20 (b)*. In this figure it is represented a maximal solution of a puzzle with plates $[1358], [1367], [1376], [1385], [1542], [1583], [1673], [1763], [1853], [2458], [2467], [2476], [2485], [2674], [2764], [2854]$. Its group has 64 elements.

6.2.3. *Figure 20 (c)*. In this figure it is represented a maximal solution of a puzzle with plates $[1257], [1275], [1286], [1465], [1478], [1564], [1682], [1752], [1874], [2356], [2387], [2783], [2653], [3457], [3486], [3754]$. Its group has 32 elements.

6.2.4. *Figure 20 (d)*. The other puzzle in Figure 20 (d) is equivalent to the puzzle in Figure 20 (c). For example, if $(13)(67)_-$ acts on the plates of one of these puzzles, it generates the plates of the other.

6.3. Puzzles « $p6$ ».

6.3.1. *Figure 21*. The non connected group is generated by

a	d	c	b	u
$(123)(45)$	$(123)(45)$	(132)	(45)	i

The plates are: $[124], [125], [134], [135], [142], [143], [145], [152], [153], [154], [234], [235], [243], [245], [253], [254], [345], [354]$.

(angle of $2\pi/3$ in the direct sense) centered in the triangles are permutations like (135) (246) and all the others generated by the translations and this one.

The group of the pattern is generated by

a	d	c	b	u
(123456)	(276435)	(127) (365)	(17) (26) (34)	(1325647)

We naturally construct triangular plates based upon these permutations of order 3: [127], [135], [143], [152], [164], [176], [237], [246], [254], [263], [365], [347], [457], [567]. In Figure 22 (b) is represented the maximal natural solution of this puzzle. Its group has 42 elements.

We fix now plate [246] in the board and look for all solutions that have associated the permutation (135) (246) to the rotation of order 3 in the middle of this triangle. There are 5 such solutions: (b)–(f).

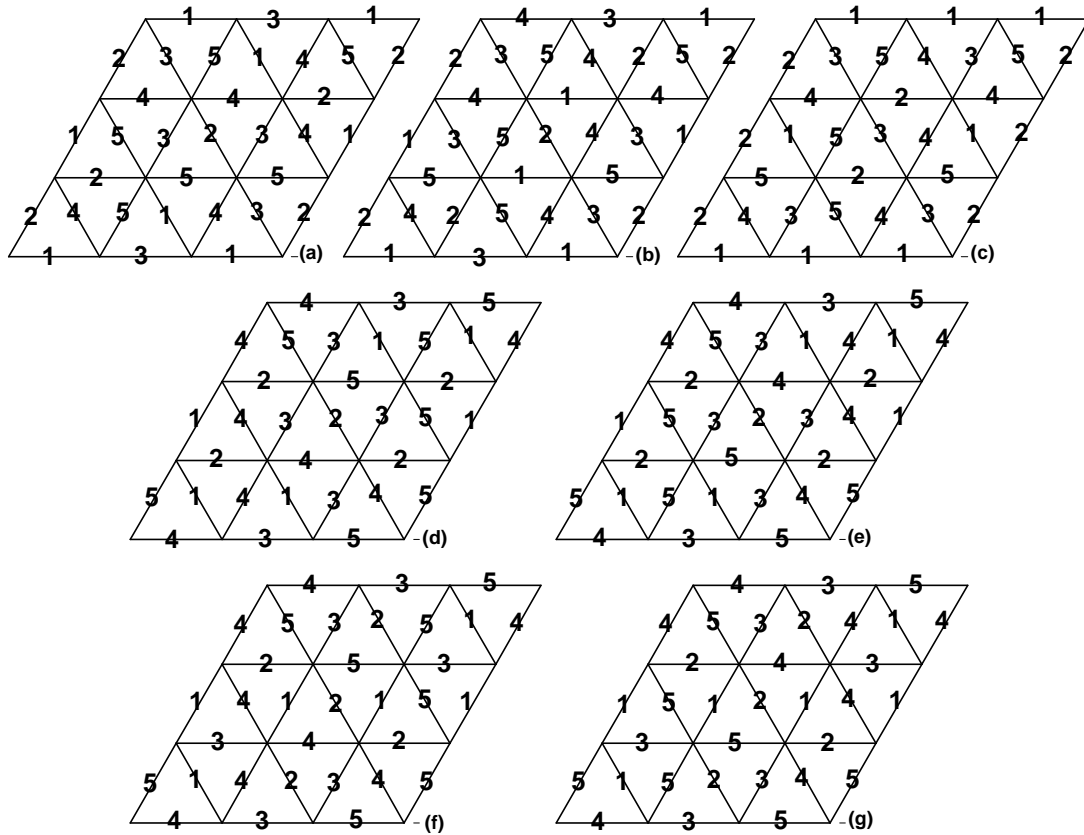


FIGURE 21.

The solutions (c) and (d) are equivalent. One can obtain (d) from (c) by operating with the permutation (123456) over the plates. Their group has no permutations of order 6, and it has 21 elements.

The solutions (e) and (f) are equivalent. One can obtain (f) from (e) by operating with the permutation (14) (25) (36) over the plates. Their group has no permutations of order 6, and it has 3 elements.

6.3.3. Figure 23. The group is generated by

a	d	c	b	u
(123456)	(156423)	(165) (243)	(14)	(25) (36)

In this figure we have a board of eight equilateral triangles as faces. The plates are [132], [135], [162], [165], [243], [246], [354], [465]. There are three equivalence classes with 2 (a) 6 (b) and 6 (c) natural solutions. Figure 23 includes representatives of the three equivalence classes.

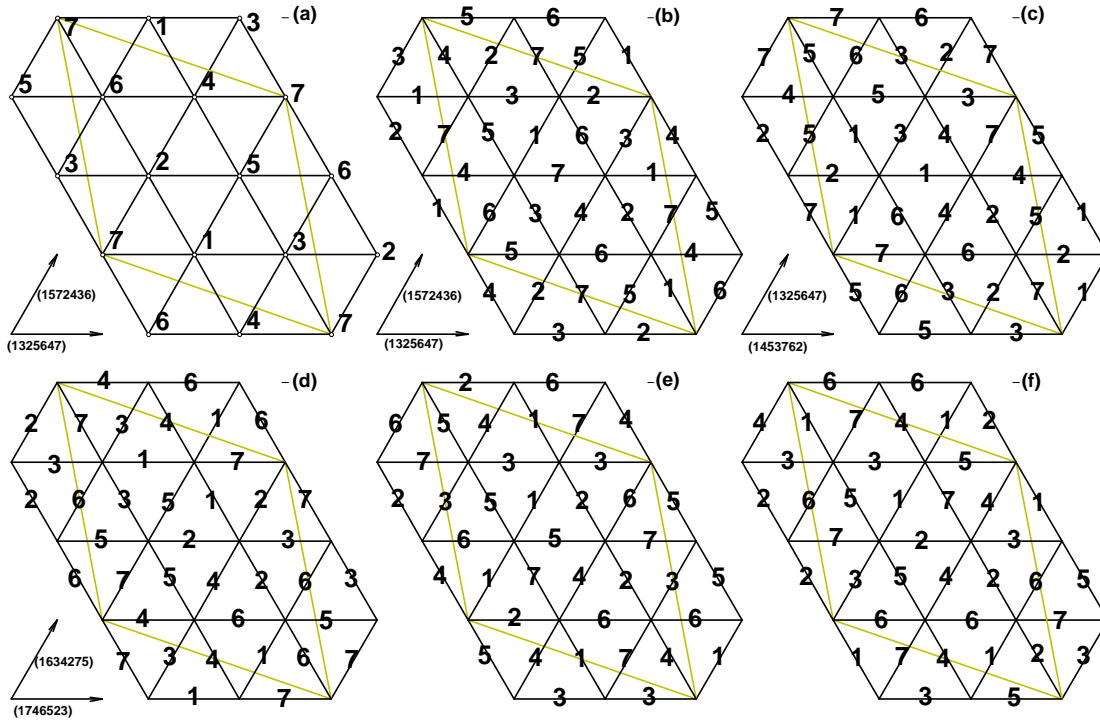


FIGURE 22.

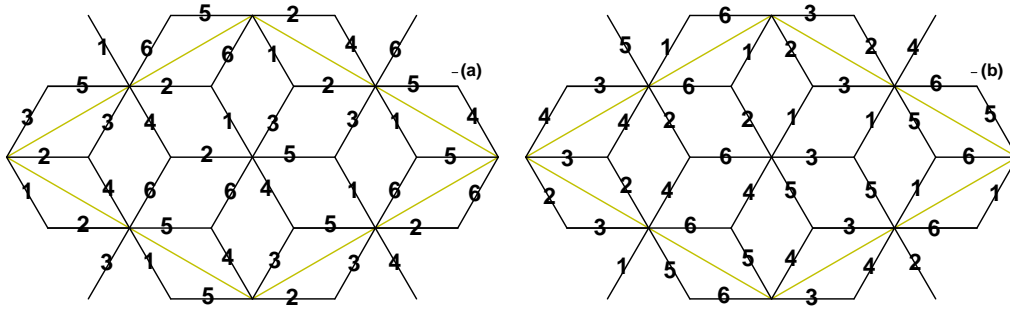


FIGURE 24.

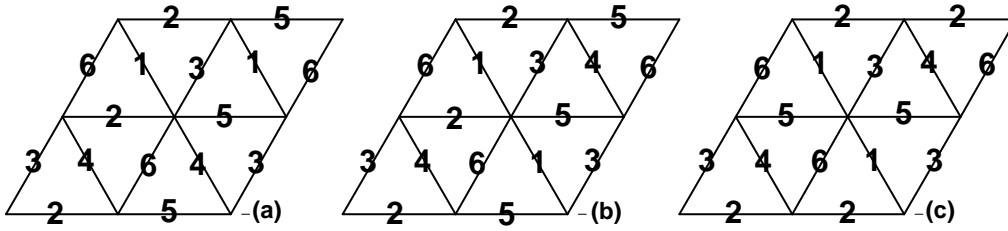


FIGURE 23.

6.3.4. *Figure 24 (a).* In this figure we have a board of twelve equilateral parallelograms and the plates are $[1316]$, $[2124]$, $[3235]$, $[4346]$, $[5154]$, $[6265]$, twice.

6.4. Puzzles « $p6mm$ ».

6.4.1. *Figure 24 (b).* The group is is generated by

a	d	x	y
(123456)	(156423)	$(14)(23)(56)$	$(15)(24)$

In this figure we have a board of twelve equilateral parallelograms and the plates are $[1313]$, $[1616]$, $[2121]$, $[2424]$, $[3232]$, $[3535]$, $[4343]$, $[4646]$, $[5151]$, $[5454]$, $[6262]$, $[6565]$.

6.4.2. *Figure 25.* In this figure, the board has 12 faces (4 regular hexagons and 8 equilateral triangles) and the plates are $[123456]$, $[153426]$, $[156423]$, $[126453]$, $[132]$, $[135]$, $[162]$, $[165]$, $[243]$, $[246]$, $[354]$, $[465]$. This board and these plates are made in such a way that one has a maximal solution with the group generated by

a	d	x	y
(123456)	(156423)	$(14)(23)(56)$	$(15)(24)$

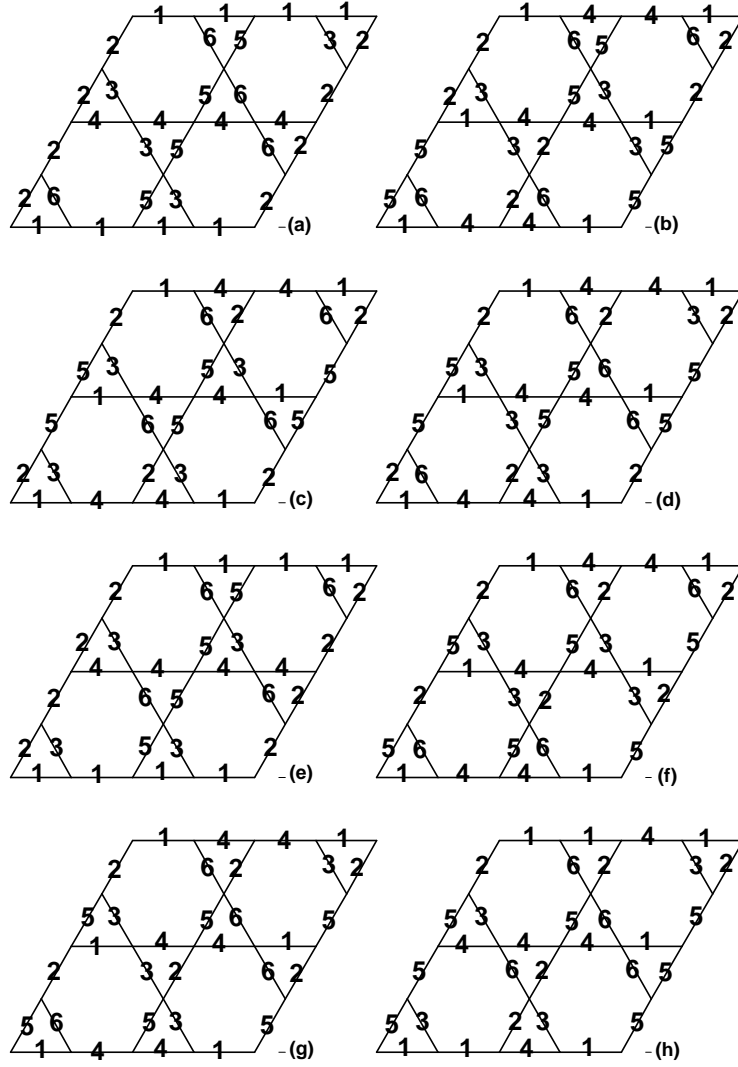


FIGURE 25.

This puzzle puzzle has 8 equivalence classes: (a) 1 has a group of order 48, (b) 1 has a group of order 24, (c)–(e) 3 have a group of order 16, (f)–(g) 2 have a group of order 8, (e) 1 has group of order 6. As $|G_P| = 48$, one has that once we put a plate over a face there are 32 different possibilities (32 natural solutions) represented in Figure 25:

$$48 \left(\frac{1}{48} + \frac{1}{24} + \frac{3}{16} + \frac{2}{8} + \frac{1}{6} \right) = 32.$$

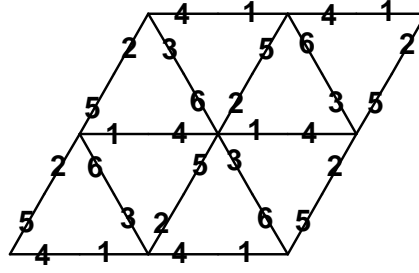


FIGURE 26.

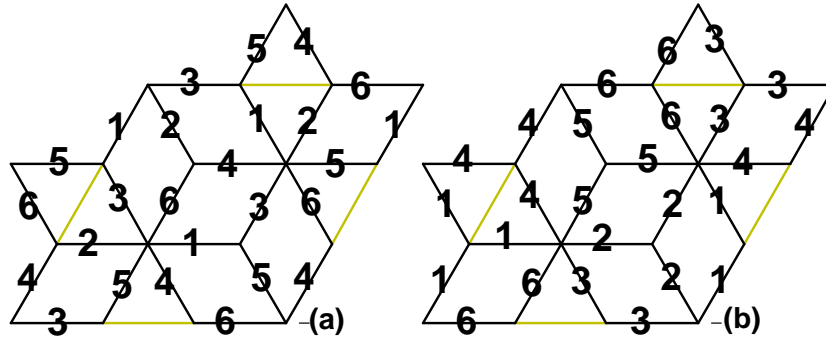


FIGURE 27.

6.4.3. *Figure 26.* The group is generated by

a	d	x	y
(123456)	(156423)	(26) (35)	(12) (36) (45)

In this figure, the board has eight faces (equilateral triangles) and the plates are $[14, 36, 25]$, $[14, 36, 52]$, $[14, 63, 25]$, $[14, 63, 52]$, $[41, 36, 25]$, $[41, 36, 52]$, $[41, 63, 25]$, $[41, 63, 52]$. Figure 26 shows a maximal solution that has all the 48 symmetries of the board.

6.4.4. *Figure 27 (a) and (b).* In Figure 27 (a) we have a board of twelve equilateral parallelograms and the plates are $[1245]$, $[1625]$, $[1465]$, $[2136]$, $[2356]$, $[3241]$, $[3461]$, $[4352]$, $[5463]$.

In Figure 27 (b) we have a board of twelve equilateral parallelograms and the plates are $[1221]$, $[1441]$, $[1661]$, $[2332]$, $[2552]$, $[3443]$, $[3663]$, $[4554]$, $[5665]$.

The group of these maximal solutions of both puzzles is generated by

a	d	x	y
(123456)	(163254)	(14) (23) (56)	(15) (24)

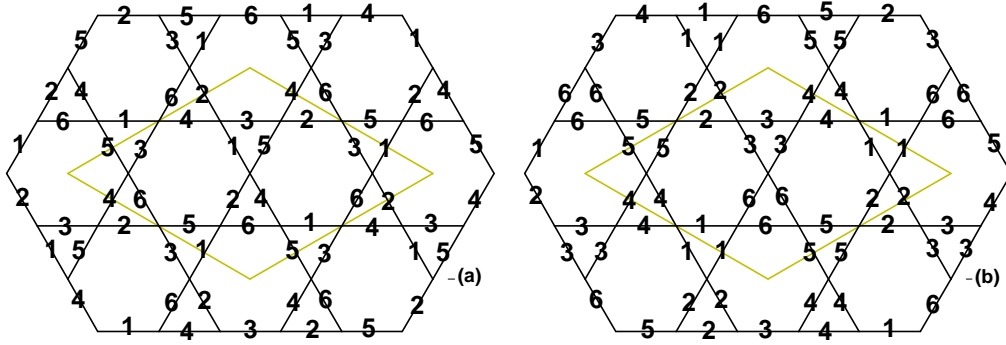


FIGURE 28.

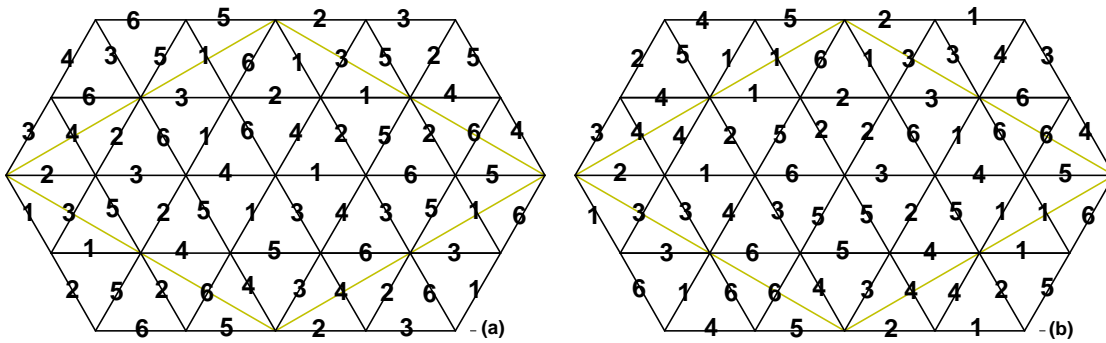


FIGURE 29.

6.4.5. *Figure 28.* The group is generated by

a	d	x	y
(123456)	(163254)	(14) (23) (56)	(15) (24)

In this figure, the board has 9 faces (3 regular hexagons and 6 equilateral triangles) and the plates are: in (a), [153], [264], three times, and [123456], [163254], [143652]; in (b), [111], [222] [333], [444] [555], [666], [123456], [163254], [143652].

6.4.6. *Figure 29 (a).* The group is generated by

a	d	x	y
(123456)	(163254)	(14) (23) (56)	(15) (24)

In this figure we have a board with 24 triangular equilateral faces.

The plates are: [153], [264], three times; [124], [125], [132], [134], [136], [145], [146], [162], [165], [235], [236], [243], [245], [256], [346], [354], [356], [465].

6.4.7. *Figure 29 (b)*. The group is generated by

a	d	x	y
(123456)	(163254)	(14) (23) (56)	(15) (24)

In this figure we have again a board with 24 triangular equilateral faces.

The plates are: [111], [222] [333], [444] [555], [666]; [124], [125], [132], [134], [136], [145], [146], [162], [165], [235], [236], [243], [245], [256], [346], [354], [356], [465].

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